

# Infinite families of regular expanders of arbitrary constant degree obtained via the modified zig-zag product

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## Abstract

The Reinhold-Vadhan-Wigderson zig-zag product construction [5] produces infinite families of good expanders of constant square degree. In this article we generalize their construction to produce infinite families of regular graphs of any constant degree. We analyze the second largest eigenvalue of this new zig-zag product to show that the modified zig-zag product of good expanders is again a good expander (yet not Ramanujan).

## 1 Introduction

Expander graphs are well connected yet sparse graphs. They have numerous applications in computer science and pure mathematics. For a wonderful introduction to expander graphs and their applications we refer the reader to [1]. Another excellent resource on the subject is the Lubotzky's book [2].

For any  $k$ -regular graph  $X$ , we denote by  $V(X)$  its vertex set and by  $E(X)$  its edge set. We denote by  $\text{Ad}(X)$  the adjacency matrix of  $X$  and let

$$\lambda(X) = \max\{|\lambda| : \lambda \text{ eigenvalue of } \text{Ad}(X), |\lambda| \neq k\}.$$

Thus,  $\lambda(X)$  is the second largest eigenvalue of  $\text{Ad}(X)$  in absolute value. If  $A$  is a collection of vertices of  $G$ , the *boundary* of  $A$ , denoted  $\partial A$ , is given by  $\partial A = \{x \in V \setminus A : \{x, y\} \in E, \text{ some } y \in A\}$ .

**Definition 1** *A finite regular graph  $X$  on  $n$  vertices and of degree  $k$  is called an  $(n, k, c)$ -expander if for every subset  $A$  of  $V(X)$  with  $|A| \leq \frac{n}{2}$  we have*

$$|\partial A| \geq c|A|.$$

*The constant  $c$  is called the expansion coefficient.*

The expansion coefficient of a regular graph  $X$  depends on  $\lambda(X)$ .

**Proposition 1** *A finite  $k$ -regular graph  $X$  on  $n$  vertices is an  $(n, k, c)$ -expander with  $2c = 1 - |\lambda(X)|/k$ .*

**Proof:** [3].

Thus, good expanders have small  $\lambda(X)$ . However, asymptotically  $\lambda(X)$  cannot be made arbitrarily small. We have the following proposition.

**Proposition 2 (Alon-Boppana)**  $\lim_{n \rightarrow \infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$ , where  $X_{n,k}$  is a  $k$ -regular graph on  $n$  vertices.

**Proof:** [3].

This upper bound on  $\lambda(G)$  leads to the following definition due to Lubotzky, Phillips and Sarnak.

**Definition 2**  $X_{n,k}$  is called a Ramanujan graph if  $\lambda(X_{n,k}) \leq 2\sqrt{k-1}$ .

Infinite families of Ramanujan graphs of constant degree have been constructed using deep results from number theory [3], [4]. The first purely combinatorial construction of infinite families of good expanders (yet not Ramanujan) of constant degree is due to Reingold, Vadhan and Wigderson [5]. These graphs are obtained by starting with two good expanders, a large graph  $G$  and a small graph  $H$ , and applying an operation called the *zig-zag product* to obtain a larger expander  $G \circledast H$ . This graph has degree equal to the square of the degree of  $H$  and its expansion coefficient is determined by the expansion coefficient of both  $G$  and  $H$ . For applications, the fact that the resulting graphs are not quite Ramanujan is irrelevant. The construction is strikingly beautiful.

In this article we introduce a modification of the zig-zag product which produces expanders of any degree. For the analysis of the eigenvalue as well as the definition of the modified zig-zag product we follow Luca Trevisan's more intuitive approach to the original construction and estimation of the second largest eigenvalue (see [6]).

## 2 The modified zig-zag product

Let  $G$  be a *regular* bipartite graph of degree  $d_1 \cdot d_2$  with  $N_1$  left vertices and  $N_2$  right vertices. (This can be obtained easily from known methods since there is no requirement on  $N_1$  and  $N_2$ ). We label the vertices of  $G$  with the integers  $1, 2, \dots, |V(G)|$ , starting with the left vertices and ending with the right vertices. We often write  $v \in V(G)$  to mean the label of  $v$ . We fix a random labeling of the edges of  $G$  near each vertex. We say that  $i$  is the  $k$ -th neighbor of  $j$  in  $G$  if  $i$  and  $j$  are adjacent and the edge connecting them is labeled  $k$  near  $j$ . Let  $H_1$  be a regular graph of degree  $d_1$  with  $d_1 \cdot d_2$  vertices and  $H_2$  be a regular graph of degree  $d_2$  with  $d_1 \cdot d_2$  vertices.

First we form the *modified replacement* product  $G \circledcirc (H_1, H_2)$  as follows. Replace each left vertex of  $G$  by a copy of  $H_1$  henceforth called a left cloud. Similarly, replace each right vertex of  $G$  by a copy of  $H_2$  henceforth called a right cloud. We denote by  $(m, n)$  the  $n$ -th vertex in the  $m$ -th cloud of  $G \circledcirc (H_1, H_2)$ . Within the clouds we maintain the edges of  $H_1$  and  $H_2$  respectively. The new graph  $G \circledcirc (H_1, H_2)$  has the following additional edges between different clouds. If  $m_1$  and  $m_2$  are adjacent vertices in  $G$  such that  $m_1$  is the  $n_2$ -nd neighbor of

$m_2$  and  $m_2$  is the  $n_1$ -st neighbor of  $m_1$ , then  $(m_1, n_1)$  and  $(m_2, n_2)$  are adjacent in  $G \textcircled{\mathfrak{r}}(H_1, H_2)$ . The graph  $G \textcircled{\mathfrak{r}}(H_1, H_2)$  is a  $(d_1, d_2)$ -biregular bipartite graph with  $(N_1 + N_2)d_1d_2$  vertices.

Using the replacement product, we form the *modified zig-zag* product  $G \textcircled{\mathfrak{z}}(H_1, H_2)$ . The graph  $G \textcircled{\mathfrak{z}}(H_1, H_2)$  has the same vertex set as the replacement product. There is an edge in  $G \textcircled{\mathfrak{z}}(H_1, H_2)$  between the vertices  $(i, p)$  and  $(j, q)$  if there are  $s$  and  $t$  such that  $\{(i, s), (i, p)\}, \{(j, t), (j, q)\}$  and  $\{(i, s), (j, t)\}$  are edges in  $G \textcircled{\mathfrak{r}}(H_1, H_2)$  (*i.e.*, if in the replacement product we can start at the vertex  $(i, p)$ , take a zig step in the  $i$ -th cloud, move to the  $j$ -th cloud and then take a zag step to reach the vertex  $(j, q)$ ). The graph  $G \textcircled{\mathfrak{z}}(H_1, H_2)$  is a  $d_1 \cdot d_2$ -regular bipartite graph on  $(N_1 + N_2)d_1d_2$  vertices. The  $N_1d_1d_2$  vertices belonging to the left clouds are the left vertices and the  $N_2d_1d_2$  vertices belonging to the right clouds are the right vertices.

To iterate, let  $G_0 = G$  and  $H_1, H_2$  be as above. Define  $G_i = G_{i-1} \textcircled{\mathfrak{z}}(H_1, H_2)$ ,  $i = 1, 2, \dots$ . Then  $\{G_i\}_{i=0}^\infty$  is an infinite family of  $d_1d_2$ -regular graphs with  $\lim_{n \rightarrow \infty} |V(G_i)| = \infty$ .

### 3 Estimation of the second largest eigenvalue of the modified zig-zag product

**Theorem 1** *Let  $G$ ,  $H_1$  and  $H_2$  be regular graphs as above. Suppose  $\lambda(G) \leq \alpha d_1d_2$ ,  $\lambda(H_1) \leq \beta_1 d_1$  and  $\lambda(H_2) \leq \beta_2 d_2$ . Then*

$$\lambda(G \textcircled{\mathfrak{z}}(H_1, H_2)) \leq \left( \alpha + \frac{\beta_1}{2} + \frac{\beta_2}{2} + \beta_1\beta_2 + \left( \beta_1^2 + \frac{\beta_1}{2} \right) \frac{d_1}{d_2} + \left( \beta_2^2 + \frac{\beta_2}{2} \right) \frac{d_2}{d_1} \right) d_1d_2.$$

**Proof:** Let  $M$  be the adjacency matrix of  $G \textcircled{\mathfrak{z}}(H_1, H_2)$ . It is a matrix of size  $N_1d_1d_2 + N_2d_1d_2$ . Then

$$\lambda(G \textcircled{\mathfrak{z}}(H_1, H_2)) = \max_{x \perp \mathbf{1}, \|x\|=1} |xMx^T|,$$

where  $\mathbf{1}$  represents the vector of the appropriate size (we will mention the size when not obvious) with all entries equal to 1.

Let  $B$  be the adjacency matrix of the graph obtained from the replacement product  $G \textcircled{\mathfrak{r}}(H_1, H_2)$  by removing the edges between clouds. If we list the vertices of  $G \textcircled{\mathfrak{z}}(H_1, H_2)$  beginning with the left vertices and ending with the right vertices, then the matrix  $B$  has the following form.

$$B = \begin{pmatrix} \text{Ad}(H_1) & & & & \\ & \ddots & & & \\ & & \text{Ad}(H_1) & & \\ & & & \text{Ad}(H_2) & \\ & & & & \ddots \\ & & & & & \text{Ad}(H_2) \end{pmatrix}.$$

Let  $A$  be the adjacency matrix of the graph obtained from the replacement product  $G\mathfrak{I}(H_1, H_2)$  by removing the edges within the clouds. With the vertices of  $G\mathfrak{Z}(H_1, H_2)$  in the same order as for  $B$ , the matrix  $A$  is a block diagonal matrix with two all zero blocks on the diagonal of sizes  $N_1 d_1 d_2 \times N_1 d_1 d_2$  and  $N_2 d_1 d_2 \times N_2 d_1 d_2$  respectively. Thus

$$A = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Then  $M = BAB$ . The largest eigenvalue of  $M$  is  $d_1 d_2$  corresponding to the eigenvector  $\mathbf{1}$ .

If  $x$  is a vector in  $\mathbb{R}^{N_1 d_1 d_2 + N_2 d_1 d_2}$ , we denote the entries of  $x$  by

$$(x_{11}, x_{12}, \dots, x_{1(d_1 d_2)}, x_{21}, x_{22}, \dots, x_{2(d_1 d_2)}, \dots, x_{(N_1 + N_2)(d_1 d_2)}).$$

Thus  $x_{ij}$  is the  $((i-1)d_1 d_2 + j)$ -th entry of  $x$ .

Given a vector  $x \in \mathbb{R}^{N_1 d_1 d_2 + N_2 d_1 d_2}$ , we define, for each  $v \in V(G)$ , a vector  $x_v \in \mathbb{R}^{d_1 d_2}$  by  $(x_v)_k := x_{vk}$ ,  $k = 1, 2, \dots, d_1 d_2$ . We decompose each vector  $x_v$  as  $x_v = x_v^{\parallel} + x_v^{\perp}$ , where  $x_v^{\parallel}$  is parallel to  $\mathbf{1}$  and  $x_v^{\perp}$  is perpendicular to  $\mathbf{1}$ . Thus,

$$x_v^{\parallel} = \left( \frac{1}{d_1 d_2} \sum_{k=1}^{d_1 d_2} x_{vk} \right) \mathbf{1} \quad \text{and} \quad x_v^{\perp} = x_v - x_v^{\parallel}. \quad (1)$$

We denote by  $e_v$  the  $v$ -th standard basis vector of  $\mathbb{R}^{N_1 + N_2}$ . Then, for  $x \in \mathbb{R}^{N_1 d_1 d_2 + N_2 d_1 d_2}$  we have

$$x = \sum_{v \in V(G)} e_v \otimes x_v = \sum_{v \in V(G)} e_v \otimes (x_v^{\parallel} + x_v^{\perp}) = \sum_{v \in V(G)} e_v \otimes x_v^{\parallel} + \sum_{v \in V(G)} e_v \otimes x_v^{\perp}.$$

We define  $x^{\parallel} := \sum_{v \in V(G)} e_v \otimes x_v^{\parallel}$  and  $x^{\perp} = \sum_{v \in V(G)} e_v \otimes x_v^{\perp}$ . Therefore

$$x = x^{\parallel} + x^{\perp}.$$

Since  $x_v^{\parallel} \perp x_v^{\perp}$  for all  $v \in V(G)$ , it follows that  $x^{\parallel} \perp x^{\perp}$  and thus  $\|x\| = \|x^{\parallel}\| + \|x^{\perp}\|$ .

With the above definitions, and using the symmetry of  $BAB$ , we have

$$|xMx^T| = |xBABx^T| = |(x^{\parallel} + x^{\perp})BAB(x^{\parallel} + x^{\perp})^T|$$

and thus

$$|xMx^T| \leq |x^{\parallel}BAB(x^{\parallel})^T| + 2|x^{\parallel}BAB(x^{\perp})^T| + |x^{\perp}BAB(x^{\perp})^T|. \quad (2)$$

Now we will find upper bounds depending on  $\alpha, \beta_1, \beta_2$  for each of the three terms above.

We start with the term  $|x^{\perp}BAB(x^{\perp})^T|$ . We have

$$|x^{\perp}BAB(x^{\perp})^T| = |x^{\perp}BA(x^{\perp}B)^T| \leq \|x^{\perp}BA\| \cdot \|x^{\perp}B\| = \|x^{\perp}B\| \cdot \|x^{\perp}B\|.$$

The first equality follows from the symmetry of  $B$ , the inequality is given by Cauchy-Schwarz and the last equality follows from the fact that  $A$  is a permutation matrix.

With the previous notation (for each  $v \in V(G)$  the vector  $x_v^{\perp}$  belongs to  $\mathbb{R}^{d_1 d_2}$ ),  $x^{\perp}B$  equals

$$(x_1^{\perp} \text{Ad}(H_1), x_2^{\perp} \text{Ad}(H_1), \dots, x_{N_1}^{\perp} \text{Ad}(H_1), x_{N_1+1}^{\perp} \text{Ad}(H_2), \dots, x_{N_1+N_2}^{\perp} \text{Ad}(H_2)).$$

For any vector  $z \in \mathbb{R}^{N_1 d_1 d_2 + N_2 d_1 d_2}$ , we denote by  $z_{H_1}$  the vector consisting of the first  $N_1 d_1 d_2$  entries of  $z$  and by  $z_{H_2}$  the vector consisting of the last  $N_2 d_1 d_2$  entries of  $z$ .

Using the fact that for each  $v \in \{1, 2, \dots, N_1\}$ ,  $x_v^{\perp}$  is a linear combination of eigenvectors for  $H_1$  different from  $\mathbf{1}$  and for each  $v \in \{N_1 + 1, N_1 + 2, \dots, N_1 + N_2\}$ ,  $x_v^{\perp}$  is a linear combination of eigenvectors for  $H_2$  different from  $\mathbf{1}$ , we have

$$\|x^{\perp}B\| \leq \beta_1 d_1 \|x_{H_1}^{\perp}\| + \beta_2 d_2 \|x_{H_2}^{\perp}\|. \quad (3)$$

Thus,

$$\|x^{\perp}B\|^2 \leq \beta_1^2 d_1^2 \|x_{H_1}^{\perp}\|^2 + 2\beta_1 \beta_2 d_1 d_2 \|x_{H_1}^{\perp}\| \cdot \|x_{H_2}^{\perp}\| + \beta_2^2 d_2^2 \|x_{H_2}^{\perp}\|^2.$$

Since

$$2\|x_{H_1}^{\perp}\| \cdot \|x_{H_2}^{\perp}\| \leq \|x_{H_1}^{\perp}\|^2 + \|x_{H_2}^{\perp}\|^2 = \|x^{\perp}\|^2$$

and  $\|x_{H_i}^{\perp}\|^2 \leq \|x^{\perp}\|^2$ ,  $i = 1, 2$ , we have

$$|x^{\perp}BAB(x^{\perp})^T| \leq \|x^{\perp}B\|^2 \leq \left( \beta_1^2 \frac{d_1}{d_2} + \beta_1 \beta_2 + \beta_2^2 \frac{d_2}{d_1} \right) d_1 d_2 \|x^{\perp}\|^2. \quad (4)$$

Next we examine the term  $|x^\perp BAB(x^\parallel)^T|$ .

Since  $x^\parallel = (x_1^\parallel, x_2^\parallel, \dots, x_{N_1}^\parallel, x_{N_1+1}^\parallel, \dots, x_{N_1+N_2}^\parallel)$ , and for each  $v \in \{1, 2, \dots, N_1\}$  the vector  $x_v^\parallel$  is an eigenvector for  $\text{Ad}(H_1)$  with eigenvalue  $d_1$  and for each  $v \in \{N_1 + 1, N_1 + 2, \dots, N_1 + N_2\}$  the vector  $x_v^\parallel$  is an eigenvector for  $\text{Ad}(H_2)$  with eigenvalue  $d_2$ , we have

$$x^\parallel B = (d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel). \quad (5)$$

Thus

$$|x^\perp BAB(x^\parallel)^T| = |x^\perp BA(x^\parallel B)^T| = |x^\perp BA(d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel)^T|$$

Since  $A$  is symmetric, we have

$$|x^\perp BAB(x^\parallel)^T| = \left| (x^\perp B) \left( (d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel) A \right)^T \right|. \quad (6)$$

Using the Cauchy-Schwartz inequality, (6) is bounded above by

$$\|x^\perp B\| \cdot \|(d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel) A\|. \quad (7)$$

Using (3) and the fact that the matrix  $A$  is a permutation matrix with the diagonal blocks consisting of zero entries, expression (7) is bounded above by

$$(\beta_1 d_1 \|x_{H_1}^\perp\| + \beta_2 d_2 \|x_{H_2}^\perp\|) \cdot (\sqrt{d_1^2 \|x_{H_1}^\parallel\|^2 + d_2^2 \|x_{H_2}^\parallel\|^2}).$$

Since

$$\sqrt{d_1^2 \|x_{H_1}^\parallel\|^2 + d_2^2 \|x_{H_2}^\parallel\|^2} \leq d_1 \|x_{H_1}^\parallel\| + d_2 \|x_{H_2}^\parallel\| \leq (d_1 + d_2) \|x^\parallel\|,$$

(7) is bounded above by

$$(\beta_1 d_1 + \beta_2 d_2) (d_1 + d_2) \|x^\perp\| \cdot \|x^\parallel\|.$$

Since  $\|x^\perp\| \cdot \|x^\parallel\| \leq \frac{1}{2} \|x\|^2$ , we have

$$|x^\perp BAB(x^\parallel)^T| \leq \frac{1}{2} \left( \beta_1 + \beta_2 + \beta_1 \frac{d_1}{d_2} + \beta_2 \frac{d_2}{d_1} \right) d_1 d_2 \|x\|^2. \quad (8)$$

Finally examine the term  $|x^\parallel BAB(x^\parallel)^T|$ .

Using (5), we have

$$|x^\parallel BAB(x^\parallel)^T| = |x^\parallel BA(x^\parallel B)^T| = \left| (d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel) A (d_1 x_{H_1}^\parallel, d_2 x_{H_2}^\parallel)^T \right| \quad (9)$$

We recall the definition (1) of  $x_v^{\parallel}$  and define  $y_v := \frac{1}{d_1 d_2} \sum_{k=1}^{d_1 d_2} x_{vk}$ ,  $v = 1, \dots, N_1, N_1 + 1, \dots, N_1 + N_2$ .

Thus (9) equals

$$\left| (d_1 y_1 \mathbf{1}, \dots, d_1 y_{N_1} \mathbf{1}, d_2 y_{N_1+1} \mathbf{1}, \dots, d_2 y_{N_1+N_2} \mathbf{1}) A (d_1 y_1 \mathbf{1}, \dots, d_1 y_{N_1} \mathbf{1}, d_2 y_{N_1+1} \mathbf{1}, \dots, d_2 y_{N_1+N_2} \mathbf{1})^T \right|,$$

where  $\mathbf{1}$  is the vector in  $\mathbb{R}^{d_1 d_2}$  with all entries 1.

By the definition of the matrix  $A$ , the first  $d_1 d_2$  entries of

$$(d_1 y_1 \mathbf{1}, \dots, d_1 y_{N_1} \mathbf{1}, d_2 y_{N_1+1} \mathbf{1}, \dots, d_2 y_{N_1+N_2} \mathbf{1}) A$$

are  $d_2 y_{N_1+j_1}, d_2 y_{N_1+j_2}, \dots, d_2 y_{N_1+j_{d_1 d_2}}$ , where  $N_1 + j_1, N_1 + j_2, \dots, N_1 + j_{d_1 d_2}$  are the neighbors of the first vertex in  $G$ . Similarly, the next  $d_1 d_2$  entries are  $d_2$  multiples of  $y$ 's corresponding to neighbors of the second vertex in  $G$ , etc. Of course, once we get to entries corresponding to neighbors of right vertices in  $G$ , these are multiples of  $d_1$ .

Therefore, the expression (9) equals

$$\left| d_1 d_2 (y_1 \sum_{j \sim 1} y_j + y_2 \sum_{j \sim 2} y_j + \dots y_{N_1+N_2} \sum_{j \sim N_1+N_2} y_j) \right|$$

$$= d_1 d_2 |(y_1, \dots, y_{N_1+N_2}) \text{Ad}(G)(y_1, \dots, y_{N_1+N_2})^T|.$$

With  $y := (y_1, \dots, y_{N_1+N_2})$  the expression above equals  $d_1 d_2 |y \text{Ad}(G) y^T|$ .

Thus, by the Cauchy-Schwartz inequality

$$|x^{\parallel} B A B (x^{\parallel})^T| \leq d_1 d_2 \|y \text{Ad}(G)\| \cdot \|y\|.$$

Since  $y \perp \mathbf{1}_{N_1 N_2}$ , we have

$$|x^{\parallel} B A B (x^{\parallel})^T| \leq d_1^2 d_2^2 \alpha \|y\|^2.$$

Since  $\|x^{\parallel}\|^2 = d_1 d_2 \|y\|^2$ , we have

$$|x^{\parallel} B A B (x^{\parallel})^T| \leq d_1 d_2 \alpha \|x^{\parallel}\|^2. \quad (10)$$

Using the bounds (4), (8) and (10), in expression (2), we obtain

$$|x M x^T| \leq \left( \beta_1^2 \frac{d_1}{d_2} + \beta_1 \beta_2 + \beta_2^2 \frac{d_2}{d_1} \right) d_1 d_2 \|x^{\perp}\|^2 + \frac{1}{2} \left( \beta_1 + \beta_2 + \beta_1 \frac{d_1}{d_2} + \beta_2 \frac{d_2}{d_1} \right) d_1 d_2 \|x\|^2 + d_1 d_2 \alpha \|x^{\parallel}\|^2.$$

Since  $\|x^{\perp}\|, \|x^{\parallel}\| \leq \|x\|$  and  $\|x\| = 1$ , we have

$$|x M x^T| \leq \left( \left( \beta_1^2 \frac{d_1}{d_2} + \beta_1 \beta_2 + \beta_2^2 \frac{d_2}{d_1} \right) + \frac{1}{2} \left( \beta_1 + \beta_2 + \beta_1 \frac{d_1}{d_2} + \beta_2 \frac{d_2}{d_1} \right) + \alpha \right) d_1 d_2,$$

which concludes the proof of the theorem.

## 4 Conclusion

Note that when  $d_1 = d_2$  the modified zig-zag product gives a slightly worse bound for  $\lambda(G \mathbin{\text{\textcircled{Z}}}(H_1, H_2))$  than the bound obtained from the zig-zag product introduced in [5]. Thus, the modified zig-zag product should only be used when constructing a family of infinite expanders of non-square degree.

## References

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